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Chapter 2: Compressible Flow

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Chapter Two – Compressible flow

Outline lecture notes

- Introduction, brief review of Thermodynamics
- The speed of sound, definition and classification of compressible flow
- Governing equations of isentropic flow with moderate area variation, stagnation properties
- Normal shock wave and oblique shock wave:
- Flow through convergent-divergent (De-Laval) nozzle
- **Fanno flow** (Adiabatic constant area duct flow with friction) and **Rayleigh flow** (Frictionless constant area duct flow with heat transfer)
- Lift and drag on supersonic airfoils

Learning Objectives

After completing this chapter, you should be able to:

- Distinguish between *incompressible and compressible flows*, and know when the approximations associated with assuming fluid incompressibility are acceptable.
- Understand some important features of different categories of compressible flows of ideal gases.
- Explain speed of sound and Mach number and their practical significance.
- Solve useful problems involving isentropic and non-isentropic flows including flows across normal shock waves.

Introduction to compressible flow

What do you mean by a compressible flow?

- Fluids have the capacity to change volume and density, i.e. **compressibility**.
- Gas is much more compressible than liquid.
- Compressible flow is a flow in which there are **significant** or **noticeable changes** in **fluid density**.
  - Just as inviscid (which means frictionless) fluids do not actually exists, incompressible fluids **do not actually exist**.
  - For example, most of the time we have treated water as an incompressible fluid, although in fact the density of sea water increases by 1% for each mile or so of depth. Hence whether or not a given flow can be considered/ treated as **incompressible is a judgment call**:

- Therefore, liquid flows will almost always be considered as **incompressible**, but gas flows could easily be **either incompressible** or **compressible**.

- Gas has large compressibility but when its velocity is low compared with the ***sonic velocity the change in density is small*** and it is then treated as an ***incompressible fluid***.

- The compressibility factor has to be considered in the following engineering problems.
  - The flight of projectiles and airplanes moving at high altitudes
  - The flow of gases through nozzles and orifices
  - In meteorological studies where the gas system involves appreciable variation in density due to greater heights involved
  - In quick-return-valves, water hammers, acoustics, rapid & repeated vibrations and such other fields.

- **Mach number, Ma** is defined as the ratio of the value of the local flow velocity, *V* to the ***speed of sound***, ***c*** *or a.*
  - The Mach number is the dominant parameter in compressible-flow analysis, with different effects depending upon its magnitude.
  - It found that, the proper criterion for a nearly incompressible flow was ***a small Mach number.***

$$Ma = \frac{V}{a} \ll 1$$

Where V = the flow velocity and

a = the speed of sound of the fluid

Flows according to Mach number:

- Incompressible — Ma < 0.3
- Subsonic — 0.3 < Ma < 0.8
- Transonic — 0.8 < Ma < 1.2
- Supersonic — 1.2 < Ma < 3.0
- Hypersonic — 3.0 < Ma

Compressible flows

- Modern aircraft are mainly powered by gas turbine engines that involve *transonic flows.*
- When *a space shuttle reenters the earth's atmosphere*, *the flow is hypersonic*. Future aircraft may be expected to operate from subsonic to hypersonic flow conditions.

In general when:

- $Ma < 1$ Subsonic
- $Ma = 1$ Sonic
- $Ma > 1$ Supersonic
- $Ma > 5$ Hypersonic

- Under small Mach number condition, **change in fluid density** are everywhere **small** in the flow field.
- If the ***density change is significant***, it follows from the ***equation of state*** the temperature and pressure change are ***also substantial***.
- The consequence ***of compressibility are not limited simply to density changes***.
- Density change means that, we can have ***significant compression or expansion work on a gas***, so the thermodynamics states of the fluid will change, meaning that in general all properties like – temperature, internal energy, entropy, enthalpy and so on can change.
- For this reason, we begin with a review of the thermodynamics need to study compressible flow.

Compressible flows require:

- Continuity equation
- Momentum equation
- Energy equation
- Equation of state

➢ **Basic thermodynamics relations**:

Mach Number: $Ma = \dfrac{U}{a}$ — a – speed of sound; U – velocity

Specific-Heat Ratio: $k = \dfrac{c_p}{c_v}$ — c_p – sp. heat at const. pressure; c_v – sp. heat at const. volume

- **Basic thermodynamics relations**:

Equation of state (perfect gas):

$$p = \rho R T$$

$$R = c_p - c_v$$

p – pressure
R – gas constant
T – temperature
ρ – density

$$a = \sqrt{k R T} \qquad c_v = \frac{R}{k-1} \qquad c_p = k c_v = \frac{k R}{k-1}$$

- Characteristic values for **air:**

$$c_p = 1005 \, \mathrm{J/(kg\,K)}$$

$$c_v = 716 \, \mathrm{J/(kg\,K)}$$

$$R = c_p - c_v = 287 \, \mathrm{J/(kg\,K)} \qquad k = \frac{c_p}{c_v} = 1.4$$

R = $\frac{R_U}{M}$

Where R_U = **8314J/(kgK) (**universal gas constant)

M = 29.98 (Molecular weight)

At standard conditions:

$$\rho = 1.2 \, \mathrm{kg/m^3} \, ; \, T = 293.15 \, \mathrm{K} \, ; \, a = 343.2 \, \mathrm{m/s}$$

Basic Thermodynamics Process:

- Isochoric process (constant volume)

$$\frac{P_1}{P_2} = \frac{T_1}{T_2} \quad Or \quad \frac{P}{T} = Constant \qquad , P_1V_1 = mRT_1 \; and \; P_2V_2 = mRT_2$$

- Isobaric process (constant pressure)

$$\frac{V_1}{V_2} = \frac{T_1}{T_2} \quad Or \quad \frac{V}{T} = Constant \qquad Mechanical\ work: W = \int_1^2 PdV = P(V_2 - V_1)$$

  - For a non –flow process: $\delta Q = \delta W + \delta E$

    Or $Q = P(V_2 - V_1)$ - $(E_2 - E_1)$ = $(E_2 + P_2V_2) - (E_1 + P_1V_1)$

  - $The\ quantity\ (E + PV) is\ a\ property\ of\ the\ system$ & $called\ enthalpy (h)$
  - $For\ an\ isobaric\ process, \delta Q = \mathrm{m}C_P(T_2 - T_1)$

- Isothermal process

$$P_1V_1 = P_2V_2 \quad or \quad PV = constant$$

Basic Thermodynamics Process:

- **Adiabatic process:**

$An\ adiabatic\ process\ follows\ the\ law$: $P_1V_1^{\gamma} = P_2V_2^{\gamma} = PV^{\gamma} = constant\ C$

$\boldsymbol{Since}$: $\delta Q = 0\ for\ an\ adiabatic\ process, then\ energy\ equation$

$$\delta W + \delta E = 0 \qquad or \quad E_1 - E_2 = \frac{P_1V_1^{\gamma} - P_2V_2^{\gamma}}{\gamma - 1}$$

- **Isentropic Process** (reversible adiabatic)

Entropy change is obtained from $T\,ds = dh - dp/\rho$

Introducing $dh = c_p\,dT$ and $p = \rho RT$

$$\int_1^2 ds = \int_1^2 c_p \frac{dT}{T} - R\int_1^2 \frac{dp}{p}$$

- Isentropic Process (reversible adiabatic)

For constant specific heat, these can be integrated to yield

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

$$= c_v \ln\frac{T_2}{T_1} + R\ln\frac{\rho_2}{\rho_1}$$

For isentropic (constant entropy) process, $s_1 = s_2$. Then

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}} = \left(\frac{\rho_2}{\rho_1}\right)^{k}$$

- The changes in the internal energy $\hat{u}$ and enthalpy h of a perfect gas are computed for constant specific heats as

$$\hat{u}_2 - \hat{u}_1 = c_v(T_2 - T_1) \qquad h_2 - h_1 = c_p(T_2 - T_1)$$

Cont…

Examples:

Argon flows through a tube such that its initial condition is $p_1 = 1.7$ MPa and $\rho_1 = 18$ kg/m^3 and its final condition is $p_2 = 248$ kPa and $T_2 = 400$ K. Estimate (*a*) the initial temperature, (*b*) the final density, (*c*) the change in enthalpy, and (*d*) the change in entropy of the gas.

Solutions:

From Table A.4 for argon, $R = 208$ m^2/(s$^2 \cdot$ K) and $k = 1.67$.

$$c_p = \frac{kR}{k-1} = \frac{1.67(208)}{1.67-1} \approx 519 \text{ m}^2/(\text{s}^2 \cdot \text{K})$$

from the ideal gas law,

$$T_1 = \frac{p_1}{\rho_1 R} = \frac{1.7 \text{ E6 N/m}^2}{(18 \text{ kg/m}^3)[208 \text{ m}^2/(\text{s}^2 \cdot \text{K})]} = 454 \text{ K}$$

$$\rho_2 = \frac{p_2}{T_2 R} = \frac{248 \text{ E3 N/m}^2}{(400 \text{ K})[208 \text{ m}^2/(\text{s}^2 \cdot \text{K})]} = 2.98 \text{ kg/m}^3$$

the enthalpy change is

$$h_2 - h_1 = c_p(T_2 - T_1) = 519(400 - 454) \approx -28{,}000 \text{ J/kg (or m}^2/\text{s}^2)$$

the entropy change is computed

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R \ln\frac{p_2}{p_1} = 519 \ln\frac{400}{454} - 208 \ln\frac{0.248 \text{ E6}}{1.7 \text{ E6}} = -66 + 400 \approx 334 \text{ m}^2/(\text{s}^2 \cdot \text{K})$$

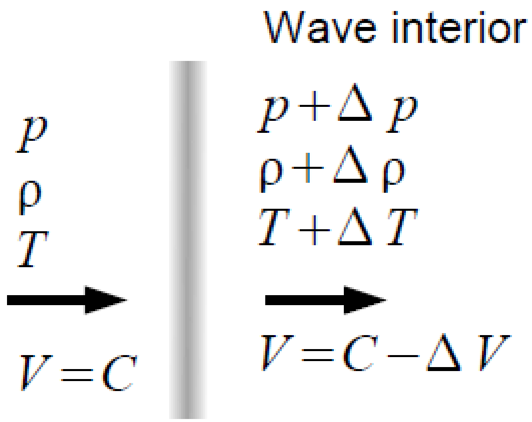
Speed of Sound

- The so-called speed of sound is the rate of propagation of a pressure pulse of infinitesimal strength through a still fluid.

- Or ***propagation speed for small pressure disturbances***

- The speed sound tells us at which speed information can be transmitted in a gaseous medium

- To derive an expression for the speed of sound(i.e. sound wave propagation) in terms of fluid property changes across the wave, let consider propagation of a sound wave of infinitesimal strength in to undisturbed medium as shown in fig. below



- Imagine a sound source that creates pressure disturbance at a certain point (i.e p is changed to P + dP due to the sound. From that point sound waves will propagate equally in all directions similar to the wave that occur when throwing a stone in to water.
- Where, C = wave speed

➢ **Note that:** the wave strength (sound wave) is regarded as a wake wave and that the change in properties are therefore infinitesimal.

➢ To derive an expression for the speed of sound, we set up the continuity equation as follows:

- **This is a 1D problem,** Continuity across the wave is:

$$\rho AC = (\rho + \Delta\rho)A(C - \Delta V) \qquad or \qquad \Delta V = C\frac{\Delta\rho}{\rho + \Delta\rho} \;\Rightarrow\; \Delta V \ll C$$

Momentum conservation along *x*:

$$\sum F_x = \dot{m}\left(V_{out} - V_{in}\right) \;\Rightarrow\; pA - (p + \Delta p)A = (\rho A C)(C - \Delta V - C)$$

$\Delta p = \rho C \Delta V$. Combining: $C^2 = \frac{\Delta p}{\Delta \rho}\left(1 + \frac{\Delta \rho}{\rho}\right)$

Combining $\Delta p = \rho C \Delta V$ and $\Delta p = \rho C \Delta V$

$$C^2 = \frac{\Delta p}{\Delta \rho}\left(1 + \frac{\Delta \rho}{\rho}\right)$$

The result indicates that the wave propagation is faster for strong disturbances (large $\Delta\rho$), *i.e.* explosion waves. For sound waves

$\Delta \rho \to 0$. then $a^2 = \frac{\partial p}{\partial \rho}$

In order to determine the derivative (slope), we need to know what kind of process is this. Given that there is no net heat exchange, it ought to be an *adiabatic* process. Since the density change is also infinitesimal it is also an *isentropic* flow. Thus:

$$a = \left(\left. \frac{\partial p}{\partial \rho} \right|_s \right)^{1/2} = \left(k \left. \frac{\partial p}{\partial \rho} \right|_T \right)^{1/2}$$

- From equation of state, $\frac{P}{\rho} = RT$

For perfect gas: $a = \sqrt{k \frac{p}{\rho}} = \sqrt{k R T}$

- From this equation the following are recognized for speed the of sound
  - ➢ The speed of sound is only dependent on gas properties and temperature
  - ➢ The higher the temperature the higher the speed of sound; this is due to increased activity of the gas molecules

Example:

- Estimate the speed of sound of carbon monoxide at 200-kPa pressure and 300°C in m/s.

Solution

From Table A.4, for CO, the molecular weight is 28.01 and k =1.40.

Thus $R_{CO} = R_u/M_m = 8314/28.01 = 297\ m^2/(s^2K)$,

For the given temperature T = 300°C + 273 = 573 K.

Thus, from the equation of speed of sound

$a_{CO} = (kRT)^{1/2} = [1.40(297)(573)]^{1/2} = 488$ m/s

Table A.4 Properties of Common Gases at 1 atm and 20°C (68°F)

Gas	Molecular weight	R, $m^2/(s^2 \cdot K)$	ρg, N/m^3	μ, $N \cdot s/m^2$	Specific-heat ratio	Power-law exponent n†
H_2	2.016	4124	0.822	9.05 E-6	1.41	0.68
He	4.003	2077	1.63	1.97 E-5	1.66	0.67
H_2O	18.02	461	7.35	1.02 E-5	1.33	1.15
Ar	39.944	208	16.3	2.24 E-5	1.67	0.72
Dry air	28.96	287	11.8	1.80 E-5	1.40	0.67
CO_2	44.01	189	17.9	1.48 E-5	1.30	0.79
CO	28.01	297	11.4	1.82 E-5	1.40	0.71
N_2	28.02	297	11.4	1.76 E-5	1.40	0.67
O_2	32.00	260	13.1	2.00 E-5	1.40	0.69
NO	30.01	277	12.1	1.90 E-5	1.40	0.78
N_2O	44.02	189	17.9	1.45 E-5	1.31	0.89
Cl_2	70.91	117	28.9	1.03 E-5	1.34	1.00
CH_4	16.04	518	6.54	1.34 E-5	1.32	0.87

Table 9.1 Sound Speed of Various Materials at 60°F (15.5°C) and 1 atm

Material	a, ft/s	a, m/s
Gases:		
H_2	4,246	1,294
He	3,281	1,000
Air	1,117	340
Ar	1,040	317
CO_2	873	266
CH_4	607	185
$^{238}UF_6$	297	91
Liquids:		
Glycerin	6,100	1,860
Water	4,890	1,490
Mercury	4,760	1,450
Ethyl alcohol	3,940	1,200
Solids:*		
Aluminum	16,900	5,150
Steel	16,600	5,060
Hickory	13,200	4,020
Ice	10,500	3,200

*Plane waves. Solids also have a *shear-wave speed*.

Adiabatic and Isentropic Steady Flow

- Applying energy equation for 1D flow:

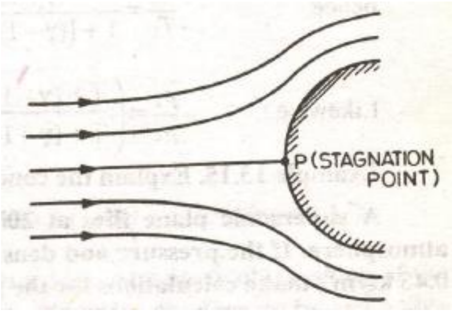
$$h_1 + \frac{V_1^2}{2} + g\,z_1 = h_2 + \frac{V_2^2}{2} + g\,z_2 - q + w_{visc}$$

- We can neglect viscous work, heat exchange and elevation (potential) energy. Then the 1D energy equation reduces to:

$$h + \frac{V^2}{2} = h_0 = \text{const}$$ where h_0 is the *stagnation enthalpy.*

for a perfect gas $h = c_p T$,

$$c_p T + \frac{V^2}{2} = c_p T_0$$ where T_0 is the *stagnation temperature.*



- $T_O = T + \frac{V^2}{2C_P}$ is the stagnation or total head temperature of the flow stream.
- Maximum velocity obtained when *enthalpy & temperature drop to(absolute) zero:*

$$V_{max} = (2h_o)^{1/2} = (2C_P T_O)^{1/2}$$

Mach number relations for adiabatic flow (isentropic or not!)

$$\text{dividing } c_p T + \frac{V^2}{2} = c_p T_0 \text{ by } c_p T$$

$$1 + \frac{V^2}{2 c_p T} = \frac{T_0}{T}, \text{ and using } c_p T = \frac{k R}{k-1} T = \frac{a^2}{k-1} \text{ we arrive at}$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} \frac{V^2}{a^2} \quad \text{or}$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} \mathrm{Ma}^2 \quad \text{and}$$

$$\frac{a_0}{a} = \left(\frac{T_0}{T} \right)^{1/2} = \left[1 + \frac{k-1}{2} \mathrm{Ma}^2 \right]^{1/2} \text{ since } a \propto T^{1/2}$$

- **Mach number relations for isentropic flow (must be adiabatic, too!)**

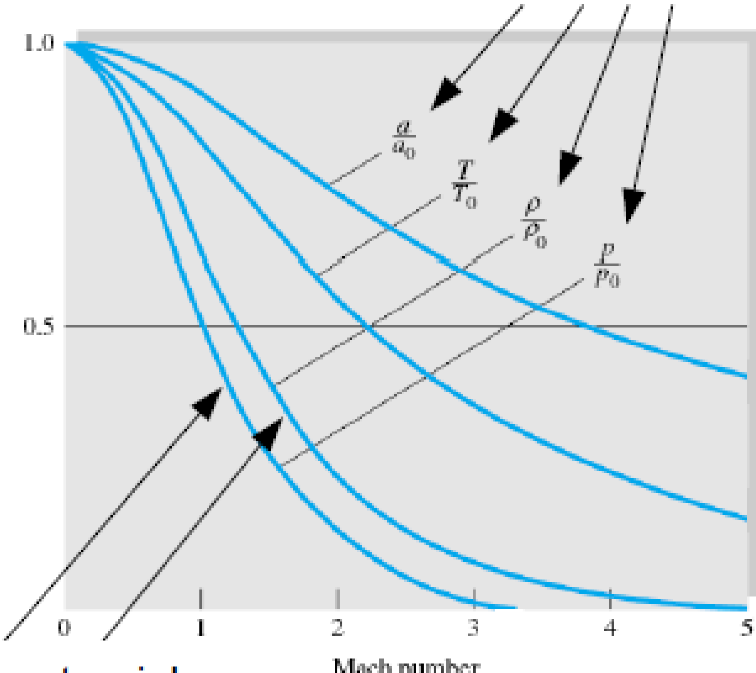
Using isentropic relations $\left(\dfrac{p_2}{p_1}\right) = \left(\dfrac{\rho_2}{\rho_1}\right)^k = \left(\dfrac{T_2}{T_1}\right)^{\frac{k}{k-1}}$ and

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}\mathrm{Ma}^2 \quad \text{(from previous slide)}$$

$$\left(\frac{p_0}{p}\right) = \left(\frac{T_0}{T}\right)^{\frac{k}{k-1}} = \left[1 + \frac{k-1}{2}\mathrm{Ma}^2\right]^{\frac{k}{k-1}}$$

$$\left(\frac{\rho_0}{\rho}\right) = \left(\frac{T_0}{T}\right)^{\frac{1}{k-1}} = \left[1 + \frac{k-1}{2}\mathrm{Ma}^2\right]^{\frac{1}{k-1}}$$

Adiabatic



Must also be isentropic!

Critical values (at Ma=1.0)

- Values at critical point, where Mach number is equal to one (sonic conditions) are of special importance for compressible flow calculations. For that reason, we mark these values by an asterisk:

$$\frac{p^*}{p_0}=\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \qquad \frac{\rho^*}{\rho_0}=\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}}$$

$$\frac{T^*}{T_0}=\left(\frac{2}{k+1}\right) \qquad \frac{a^*}{a_0}=\left(\frac{2}{k+1}\right)^{1/2}$$

$$V^*=a^*=\sqrt{kRT^*}=\left(\frac{2k}{k+1}RT_0\right)^{1/2}$$

For air ($k = 1.4$) these ratios have the following values:

$$\frac{p^*}{p_0}=0.5283\,; \quad \frac{\rho^*}{\rho_0}=0.6339\,; \quad \frac{T^*}{T_0}=0.8333\,; \quad \frac{a^*}{a_0}=0.9129$$

Some Useful Numbers for Air

- For $k = 1.4$, the following numerical versions of the isentropic and adiabatic flow formulas are obtained:

$$\frac{T_0}{T} = 1 + 0.2\ \mathrm{Ma}^2 \qquad \frac{\rho_0}{\rho} = (1 + 0.2\ \mathrm{Ma}^2)^{2.5}$$

$$\frac{p_0}{p} = (1 + 0.2\ \mathrm{Ma}^2)^{3.5}$$

- Or, if we are given the properties, it is equally easy to solve for the Mach number (again with $k = 1.4$)

$$\mathrm{Ma}^2 = 5\left(\frac{T_0}{T} - 1\right) = 5\left[\left(\frac{\rho_0}{\rho}\right)^{2/5} - 1\right] = 5\left[\left(\frac{p_0}{p}\right)^{2/7} - 1\right]$$

- **<u>Note that</u>** these isentropic-flow formulas serve *as the equivalent of the frictionless adiabatic momentum and energy equations*. They relate velocity to physical properties for a perfect gas, but they are *not* the "solution" to a gas-dynamics problem.

Example:1. Explain the concept of stagnation properties:

- *A supersonic plane flies at 2000km/hr at an altitude of 9km above sea level in standard atmospheric. If the pressure and density of air at this attitude are states to be 30kN/m^2 absolute and 0.45kg/m^3, make calculations for the pressure, temperature and density at the stagnation point on the nose of the plane. Take R = 287 J/kg.K and γ= 1.4*

Solution : Sonic velocity $a = \sqrt{\gamma RT} = \sqrt{\gamma (p/\rho)} = \sqrt{1.4 \times 30 \times 10^3 / 0.45} = 305.5 \text{ m/s}$

$$\text{Speed of plane } V = 2000 \text{ km/hr} = \frac{2000 \times 10^3}{3600} = 555.56 \text{ m/s}$$

$$\text{Mach number } M = \frac{V}{a} = \frac{555.56}{305.5} = 1.818$$

From characteristic gas equation, $\frac{p}{\rho} = RT$

$$\text{Temperature } T = \frac{p}{\rho R} = \frac{30 \times 10^3}{0.45 \times 287} = 233.3 \text{ K}$$

$$\text{Stagnation pressure, } p_0 = p\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

$$= 30\left(1 + \frac{1.4 - 1}{2} \times 1.818^2\right)^{\frac{1.4}{1.4 - 1}} = \mathbf{177.19\ kN/m^2}$$

$$\text{Stagnation temperature, } T_0 = T\left[1 + \frac{\gamma - 1}{2} M^2\right] = 233.3\left[1 + \frac{1.4 - 1}{2} \times 1.818^2\right] = \mathbf{387.5\ K}$$

$$\text{Stagnation density, } \rho_0 = \frac{p_0}{RT_0} = \frac{177.19 \times 10^3}{287 \times 387.5} = \mathbf{1.583\ kg/m^3}$$

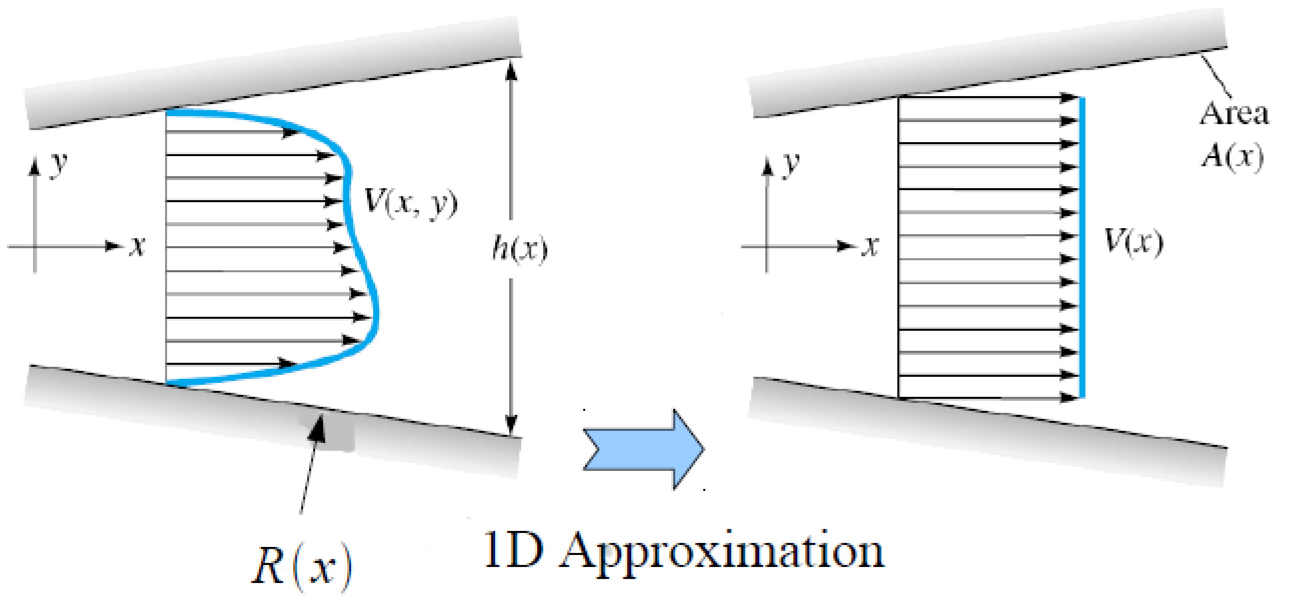
EXAMPLE 9.3

Air flows adiabatically through a duct. At point 1 the velocity is 240 m/s, with T_1 = 320 K and p_1 = 170 kPa. Compute (*a*) T_0, (*b*) p_{01}, (*c*) ρ_0, (*d*) Ma, (*e*) V_{max}, and (*f*) V^*. At point 2 further downstream V_2 = 290 m/s and p_2 = 135 kPa. (*g*) What is the stagnation pressure p_{02}?

Isentropic Flow with Area Changes

- By combining the isentropic- and/or adiabatic-flow relations with the equation of continuity we can study practical compressible-flow problems. This section treats the one dimensional flow approximation.

One-dimensional Flow in Ducts



(a) real-fluid velocity profile; *(b) one-dimensional approximation*.

$$\text{Valid for } \frac{dh}{dx} \ll 1 \text{ and } h(x) \ll R(x)$$

❑ Basic Velocity and Pressure Changes

From the differential form of Mass Conservation

$$\frac{d\rho}{\rho}+\frac{dV}{V}+\frac{dA}{A}=0$$

and Momentum Conservation

$$\frac{dp}{\rho}+V\,dV=0$$

using the speed of sound in differential form: $dp=a^2 d\rho$

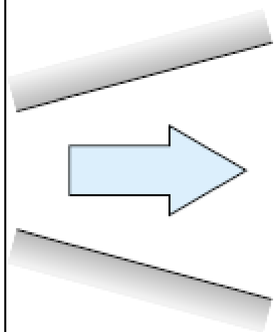
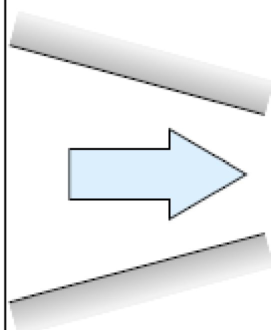
after rearranging, $\frac{dV}{V}=\frac{dA}{A}\frac{1}{\mathrm{Ma}^2-1}=-\frac{dp}{\rho V^2}$ ***How ? proof***

- An appropriate equation of motion in the streamwise direction *for the steady*, *one-dimensional*, *and isentropic* (adiabatic and frictionless) flow of *an ideal gas is obtained* from Eq. 11.41 as
- ***See Munson page 593 -594*** $dp+\frac{1}{2}\rho\, d(V^2)+\gamma\, dz=0 \Rightarrow \frac{dp}{\rho V^2}=-\frac{dV}{V}$

Convergent (nozzle) and Divergent (diffuser) Duct

Converging: $dA < 0$
Diverging: $dA > 0$

$$\frac{dV}{V} = \frac{dA}{A}\frac{1}{\mathrm{Ma}^2 - 1} = \frac{-dp}{\rho V^2}$$

Duct geometry		*Subsonic* Ma < 1	*Supersonic* Ma > 1
	$dA > 0$	$dV < 0$ $dp > 0$ Subsonic diffuser	$dV > 0$ $dp < 0$ Supersonic nozzle
	$dA < 0$	$dV > 0$ $dp < 0$ Subsonic nozzle	$dV < 0$ $dp > 0$ Supersonic diffuser

Subsonic Diverging Duct:
Velocity decreases
Pressure increases

Supersonic Diverging Duct:
Velocity increases
Pressure decreases

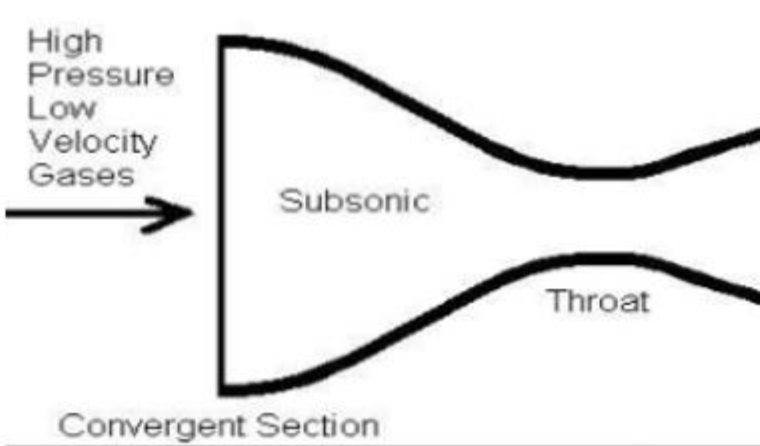
Subsonic Converging Duct:
Velocity increases
Pressure decreases

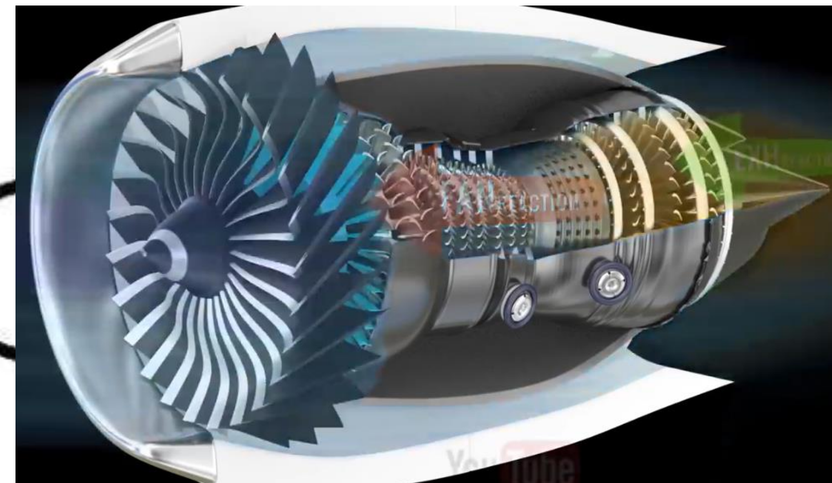
Supersonic Converging Duct:
Velocity decreases
Pressure increases

- ***Flow through converging–diverging nozzle (nozzle with a throat)***

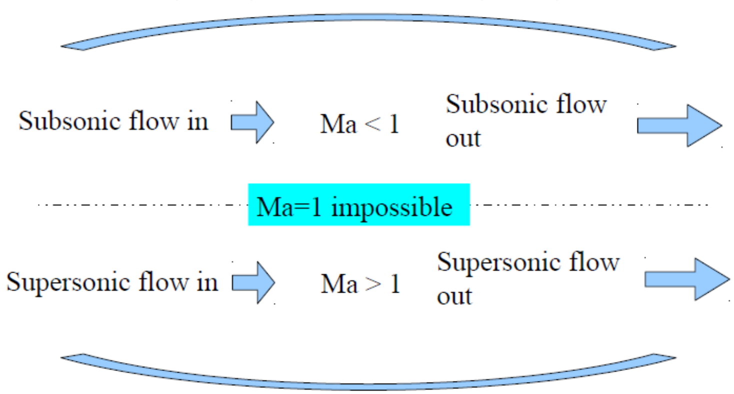
  (De Laval nozzle)

  - It is required to accelerate a flow from subsonic to supersonic flow conditions.
  - Used in supersonic aircrafts





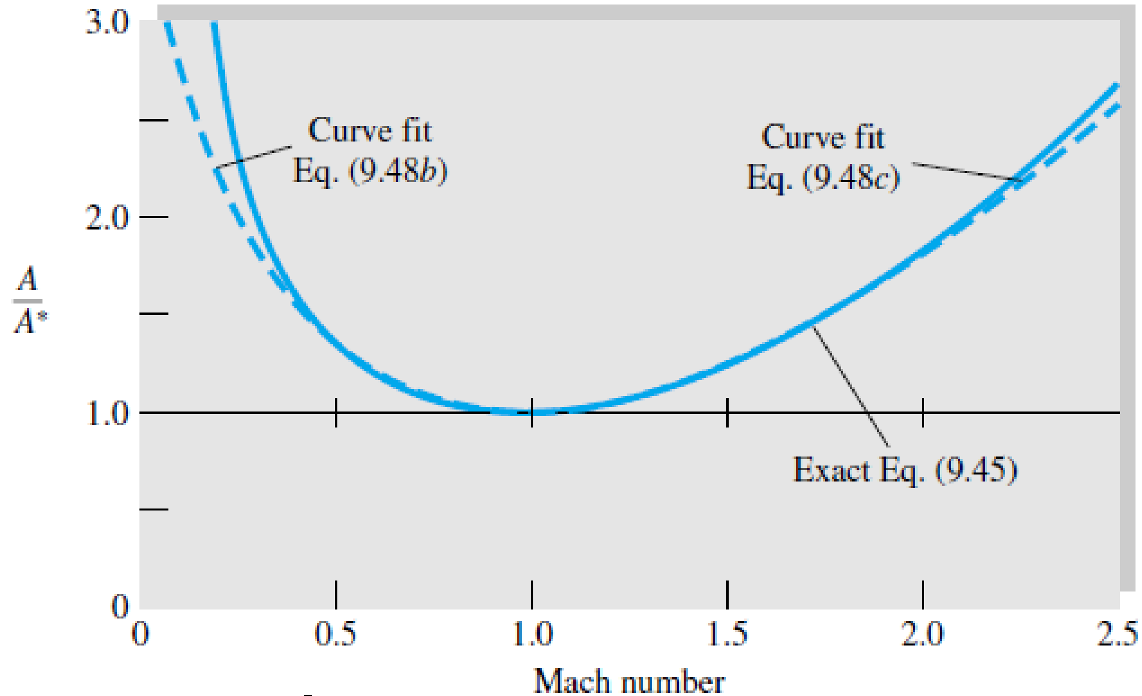
- ***Flow through diverging - converging nozzle (nozzle with a bulge)***



Perfect Gas Area Change

- Area ratio (relative to the critical area) can be expressed for the convergent/ divergent nozzle as a function of Mach number only. After lengthy algebra:

$$\frac{A}{A^*}=\frac{1}{\mathrm{Ma}}\left[\frac{1+\frac{k-1}{2}\mathrm{Ma}^2}{\frac{k+1}{2}}\right]^{\frac{1}{2}\frac{k+1}{k-1}}$$



For, k = 1.4

$$\frac{A}{A^*}=\frac{1}{\mathrm{Ma}}\frac{(1+0.2\,\mathrm{Ma}^2)^3}{1.728}$$

- **Choked flow** occurs when the Mach number is 1.0 at the minimum cross-sectional area.
- From the 1D mass conservation, *the ratio of mass flow rates per unit area* is inversely proportional *to the area ratio*:

$$\frac{\rho V}{\rho^* V^*} = \frac{A^*}{A}$$

$$\dot{m}_{\max} = \rho^* A^* V^* = \rho_0 \left(\frac{2}{k+1}\right)^{1/(k-1)} A^* \left(\frac{2k}{k+1} R T_0\right)^{1/2}$$

$$= k^{1/2} \left(\frac{2}{k+1}\right)^{(1/2)(k+1)/(k-1)} A^* \rho_0 (R T_0)^{1/2}$$

For $k = 1.4$ this reduces to

$$\dot{m}_{\max} = 0.6847 A^* \rho_0 (R T_0)^{1/2} = \frac{0.6847 p_0 A^*}{(R T_0)^{1/2}}$$

Ma

- The maximum flow rate occurs when the narrowest area (the throat) reaches the sonic conditions (Ma=1). *We call such flow choked*, since any increase of incoming velocity will *decrease* the mass flow rate. The maximum mass flow rate is thus:

$$\boxed{\dot{m}_{max} = \rho^* A^* V^*}$$

The Local Mass Flow Function

- Here is a useful mass flow function, giving the actual mass flow rate when the flow ***is not choked***, as a function of *local* area, *A*, local pressure, *p*, and stagnation pressure and temperature:

$$\text{Mass-flow function} = \frac{\dot{m}}{A}\frac{\sqrt{RT_0}}{p_0} = \sqrt{\frac{2k}{k-1}\left(\frac{p}{p_0}\right)^{2/k}\left[1-\left(\frac{p}{p_0}\right)^{(k-1)/k}\right]}$$

- The mass flow function starts at 0, for $p=p_0$, and levels off as p/p_0 approaches the critical value of 0.5283 (sonic conditions).

Isentropic Flow Tables for k = 1.4 ...cont'd

Ma	p/p_0	ρ/ρ_0	T/T_0	A/A^*
0.0	1.0	1.0	1.0	∞
0.02	0.9997	0.9998	0.9999	28.9421
0.04	0.9989	0.9992	0.9997	14.4815
0.06	0.9975	0.9982	0.9993	9.6659
0.08	0.9955	0.9968	0.9987	7.2616
0.1	0.9930	0.9950	0.9980	5.8218
0.12	0.9900	0.9928	0.9971	4.8643
0.14	0.9864	0.9903	0.9961	4.1824
0.16	0.9823	0.9873	0.9949	3.6727
0.18	0.9776	0.9840	0.9936	3.2779
0.2	0.9725	0.9803	0.9921	2.9635
0.22	0.9668	0.9762	0.9904	2.7076
0.24	0.9607	0.9718	0.9886	2.4956
0.26	0.9541	0.9670	0.9867	2.3173
0.28	0.9470	0.9619	0.9846	2.1656
0.3	0.9395	0.9564	0.9823	2.0351
0.32	0.9315	0.9506	0.9799	1.9219
0.34	0.9231	0.9445	0.9774	1.8229
0.36	0.9143	0.9380	0.9747	1.7358
0.38	0.9052	0.9313	0.9719	1.6587
0.4	0.8956	0.9243	0.9690	1.5901
0.42	0.8857	0.9170	0.9659	1.5289
0.44	0.8755	0.9094	0.9627	1.4740
0.46	0.8650	0.9016	0.9594	1.4246
0.48	0.8541	0.8935	0.9559	1.3801
0.5	0.8430	0.8852	0.9524	1.3398
0.52	0.8317	0.8766	0.9487	1.3034
0.54	0.8201	0.8679	0.9449	1.2703
0.56	0.8082	0.8589	0.9410	1.2403
0.58	0.7962	0.8498	0.9370	1.2130
0.6	0.7840	0.8405	0.9328	1.1882
0.62	0.7716	0.8310	0.9286	1.1656
0.64	0.7591	0.8213	0.9243	1.1451
0.66	0.7465	0.8115	0.9199	1.1265
0.68	0.7338	0.8016	0.9153	1.1097
0.7	0.7209	0.7916	0.9107	1.0944
0.72	0.7080	0.7814	0.9061	1.0806

Isentropic Flow Table (continued)

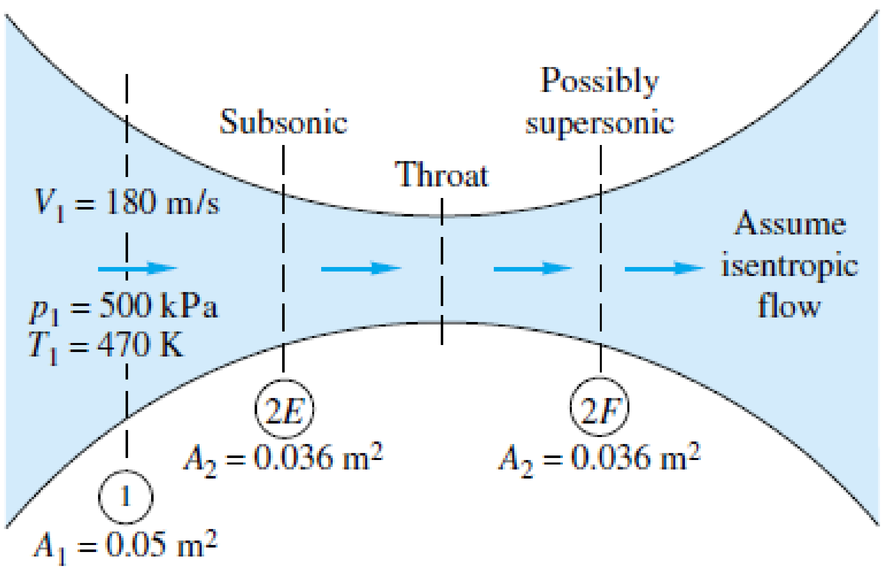
Ma	p/p_0	ρ/ρ_0	T/T_0	A/A^*
0.74	0.6951	0.7712	0.9013	1.0681
0.76	0.6821	0.7609	0.8964	1.0570
0.78	0.6690	0.7505	0.8915	1.0471
0.8	0.6560	0.7400	0.8865	1.0382
0.82	0.6430	0.7295	0.8815	1.0305
0.84	0.6300	0.7189	0.8763	1.0237
0.86	0.6170	0.7083	0.8711	1.0179
0.88	0.6041	0.6977	0.8659	1.0129
0.9	0.5913	0.6870	0.8606	1.0089
0.92	0.5785	0.6764	0.8552	1.0056
0.94	0.5658	0.6658	0.8498	1.0031
0.96	0.5532	0.6551	0.8444	1.0014
0.98	0.5407	0.6445	0.8389	1.0003
1.0	0.5283	0.6339	0.8333	1.0000
1.02	0.5160	0.6234	0.8278	1.0003
1.04	0.5039	0.6129	0.8222	1.0013
1.06	0.4919	0.6024	0.8165	1.0029
1.08	0.4800	0.5920	0.8108	1.0051
1.1	0.4684	0.5817	0.8052	1.0079
1.12	0.4568	0.5714	0.7994	1.0113
1.14	0.4455	0.5612	0.7937	1.0153
1.16	0.4343	0.5511	0.7879	1.0198
1.18	0.4232	0.5411	0.7822	1.0248
1.2	0.4124	0.5311	0.7764	1.0304
1.22	0.4017	0.5213	0.7706	1.0366
1.24	0.3912	0.5115	0.7648	1.0432
1.26	0.3809	0.5019	0.7590	1.0504
1.28	0.3708	0.4923	0.7532	1.0581
1.3	0.3609	0.4829	0.7474	1.0663
1.32	0.3512	0.4736	0.7416	1.0750
1.34	0.3417	0.4644	0.7358	1.0842
1.36	0.3323	0.4553	0.7300	1.0940
1.38	0.3232	0.4463	0.7242	1.1042
1.4	0.3142	0.4374	0.7184	1.1149
1.42	0.3055	0.4287	0.7126	1.1262
1.44	0.2969	0.4201	0.7069	1.1379
1.46	0.2886	0.4116	0.7011	1.1501

EXAMPLE 9.4

Air flows isentropically through a duct. At section 1 the area is 0.05 m^2 and $V_1 = 180$ m/s, $p_1 = 500$ kPa, and $T_1 = 470$ K. Compute (*a*) T_0, (*b*) Ma_1, (*c*) p_0, and (*d*) both A^* and $\dot{m}$. If at section 2 the area is 0.036 m^2, compute Ma_2 and p_2 if the flow is (*e*) subsonic or (*f*) supersonic. Assume $k = 1.4$.

Solution

Part (a) A general sketch of the problem is shown in Fig. E9.4. With V_1 and T_1 known, the energy equation (9.23) gives



E9.4

$$T_0 = T_1 + \frac{V_1^2}{2c_p} = 470 + \frac{(180)^2}{2(1005)} = 486 \text{ K} \qquad \textit{Ans. (a)}$$

Part (b) The local sound speed $a_1 = \sqrt{kRT_1} = [(1.4)(287)(470)]^{1/2} = 435$ m/s. Hence

$$\mathrm{Ma}_1 = \frac{V_1}{a_1} = \frac{180}{435} = 0.414 \qquad \textit{Ans. (b)}$$

Part (c) With Ma_1 known, the stagnation pressure follows from Eq. (9.34):

$$p_0 = p_1(1 + 0.2\,\mathrm{Ma}_1^2)^{3.5} = (500\text{ kPa})[1 + 0.2(0.414)^2]^{3.5} = 563\text{ kPa} \qquad \textit{Ans. (c)}$$

Part (d) Similarly, from Eq. (9.45), the critical sonic-throat area is

$$\frac{A_1}{A^*} = \frac{(1 + 0.2\,\mathrm{Ma}_1^2)^3}{1.728\,\mathrm{Ma}_1} = \frac{[1 + 0.2(0.414)^2]^3}{1.728(0.414)} = 1.547$$

or

$$A^* = \frac{A_1}{1.547} = \frac{0.05\text{ m}^2}{1.547} = 0.0323\text{ m}^2 \qquad \textit{Ans. (d)}$$

This throat must *actually be present* in the duct if the flow is to become supersonic.

We now know A^*. So to compute the mass flow we can use Eq. (9.46), which remains valid, based on the numerical value of A^*, whether or not a throat actually exists:

$$\dot{m} = 0.6847\frac{p_0 A^*}{\sqrt{RT_0}} = 0.6847\frac{(563{,}000)(0.0323)}{\sqrt{(287)(486)}} = 33.4\text{ kg/s} \qquad \textit{Ans. (d)}$$

Or we could fare equally well with our new "local mass flow" formula, Eq. (9.47), using, say, the pressure and area at section 1. Given $p_1/p_0 = 500/563 = 0.889$, Eq. (9.47) yields

$$\dot{m}\frac{\sqrt{287(486)}}{563{,}000(0.05)} = \sqrt{\frac{2(1.4)}{0.4}(0.889)^{2/1.4}[1 - (0.889)^{0.4/1.4}]} = 0.447 \qquad \dot{m} = 33.4\,\frac{\text{kg}}{\text{s}} \quad \textit{Ans. (d)}$$

Normal Shocks

- When air undergoes large and rapid compression (e.g. following an explosion, the release of engine gases into an exhaust pipe, or where an aircraft or a bullet flies at supersonic velocity) a thin wave of large pressure change is produced as shown in Fig.1.

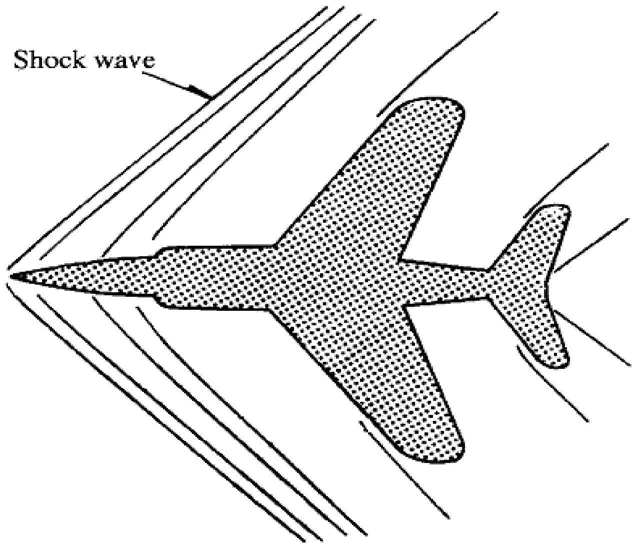
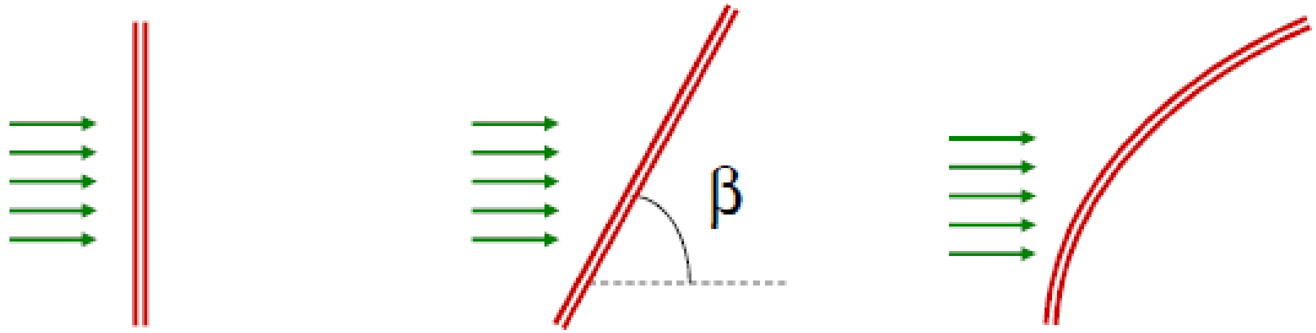


Fig. 1, Jet plane flying at supersonic velocity

Definition of shock wave

- Shock wave is a very thin *(a few micrometers thick)*region in a flow where a supersonic flow is decelerated to subsonic flow. The process is adiabatic but non-isentropic.

Types of Shock Waves:

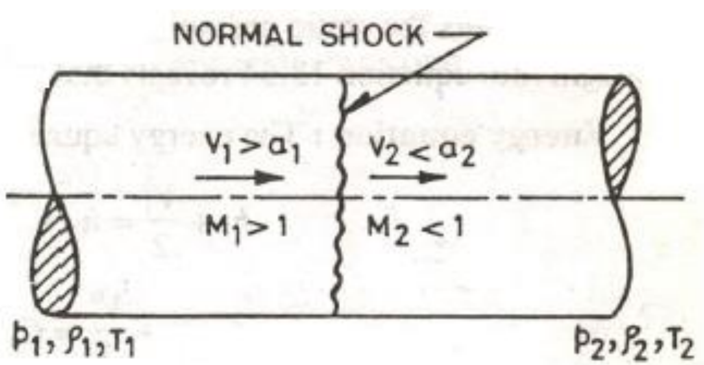


Normal shock wave
- easiest to analyze

Oblique shock wave
- will be analyzed based on normal shock relations

Curved shock wave
- difficult & will not be analyzed in this class

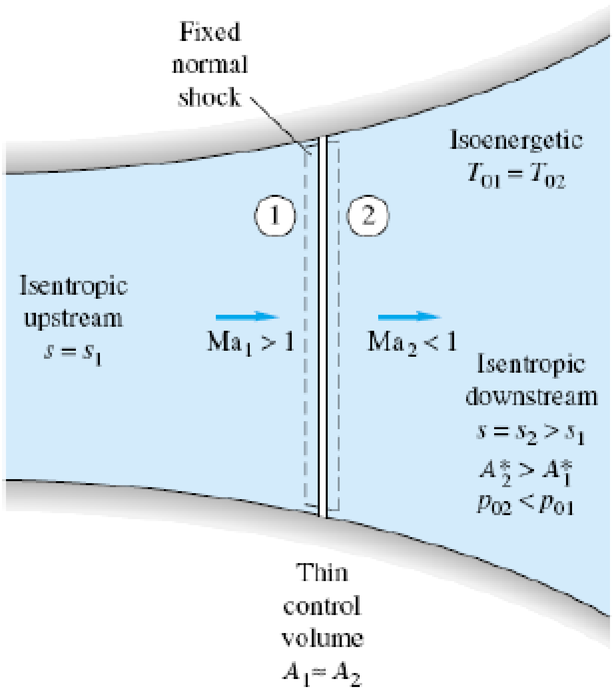
- **Normal shock**: plane of shock perpendicular to the direction of flow



ii) Oblique shock: plane of shock inclined to the direction of flow

Normal-Shock Wave Relations:

- To compute all property changes rather than just the wave speed, we use all our basic one-dimensional steady-flow relations.



Conservation laws (1D):

Mass: $\rho_1 V_1 = \rho_2 V_2 = G = const$

Momentum: $p_1 - p_2 = \rho_2 V_2^2 - \rho_1 V_1^2$

Energy: $h_1 + \dfrac{V_1^2}{2} = h_2 + \dfrac{V_2^2}{2} = h_0 = const$

+ the ideal gas relations

Equation of state: $\dfrac{p_1}{\rho_1 T_1} = \dfrac{p_2}{\rho_2 T_2}$

Constant c_p: $h = c_p T$; $k = const$

The system is closed: 5 equations for 5 unknowns

- The first successful analyses of these ***normal-shock relations*** are credited to W. J. M. Rankine (1870) and A. Hugoniot (1887), hence the modern term ***Rankine-Hugoniot relations***.

  - Assumed upstream conditions (p_1, V_1, ρ_1, h_1, T_1) and (p_2, V_2, ρ_2, h_2, T_2) are unknowns
  - Due to velocity-squared term, two solutions are found, and from 2nd law of thermodynamic $S_2 > S_1$ and again with eliminating V_1 and V_2 from mass & energy equations the ***Rankine-Hugoniot relations obtained as:***

$$h_2 - h_1 = \frac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right)$$

- Introducing the perfect-gas law ***h =C$_p$T = kp/[(k -1)ρ],*** *Pressure and density ratios* are :

$$\frac{P_2}{P_1}=\frac{\left[\left(\frac{\gamma+1}{\gamma-1}\right)\frac{\rho_2}{\rho_1}-1\right]}{\left[\left(\frac{\gamma+1}{\gamma-1}\right)-\frac{\rho_2}{\rho_1}\right]} \quad \textbf{or} \quad \frac{\rho_2}{\rho_1}=\frac{\left[\left(\frac{\gamma+1}{\gamma-1}\right)\frac{P_2}{P_1}+1\right]}{\left[\left(\frac{\gamma+1}{\gamma-1}\right)+\frac{P_2}{P_1}\right]}$$

From conservation of mass: $\frac{\rho_2}{\rho_1}=\frac{V_1}{V_2}$ ⇒ $\frac{\rho_2}{\rho_1}=\frac{1+\beta p_2/p_1}{\beta+p_2/p_1}$ $\quad \beta=\frac{k+1}{k-1}$

From equation of state: $\frac{T_2}{T_1}=\frac{P_2}{P_1}\frac{\rho_1}{\rho_2}$

- Also, the actual change in entropy across the shock can be computed from the perfect gas relation:

$$\frac{s_2-s_1}{c_v}=\ln\left[\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^k\right]$$

- Property change across the normal shock for a perfect gas are obtained as:

$$\frac{p_2}{p_1}=\frac{1}{k+1}\left[2k\,\mathrm{Ma}_1^2-(k-1)\right]$$

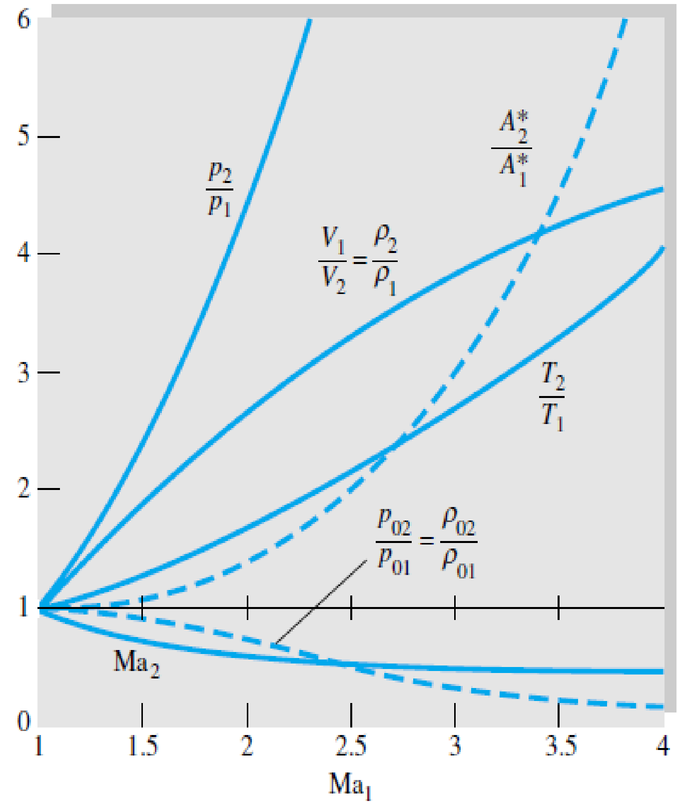
$$\mathrm{Ma}_2^2=\frac{(k-1)\mathrm{Ma}_1^2+2}{2k\,\mathrm{Ma}_1^2-(k-1)}$$

$$\frac{\rho_2}{\rho_1}=\frac{(k+1)\mathrm{Ma}_1^2}{(k-1)\mathrm{Ma}_1^2+2}=\frac{V_1}{V_2}$$

$$\frac{T_2}{T_1}=\left[2+(k-1)\mathrm{Ma}_1^2\right]\frac{2k\,\mathrm{Ma}_1^2-(k-1)}{(k+1)^2\mathrm{Ma}_1^2}$$

$$\frac{p_{02}}{p_{01}}=\frac{\rho_{02}}{\rho_{01}}=\left[\frac{(k+1)\mathrm{Ma}_1^2}{2+(k-1)\mathrm{Ma}_1^2}\right]^{\frac{k}{(k-1)}}\left[\frac{k+1}{2k\,\mathrm{Ma}_1^2-(k-1)}\right]^{\frac{1}{k-1}}$$

$$\frac{A_2^*}{A_1^*}=\frac{\mathrm{Ma}_2}{\mathrm{Ma}_1}\left[\frac{2+(k-1)\,\mathrm{Ma}_1^2}{2+(k-1)\,\mathrm{Ma}_2^2}\right]^{(1/2)(k+1)/(k-1)}$$



Change in flow properties Across a normal-shock wave for k = 1.4.

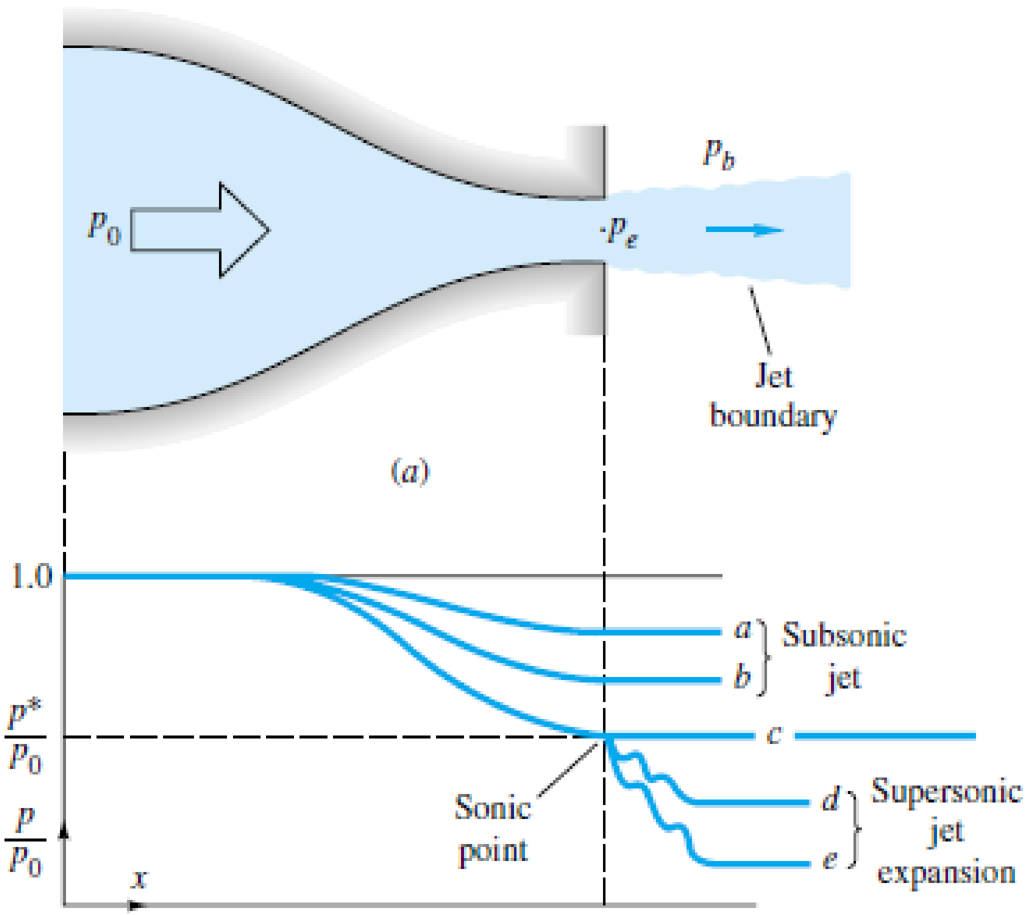
Some remarks

- The upstream flow is supersonic, and the downstream flow is subsonic.
- There is entropy increase across the wave, thus the flow is adiabatic but **not** isentropic *(because it is irreversible)*. So, $T_{01} = T_{02}$ and $P_{01} \neq P_{02}$.
- Shock wave is very thin – typically few millimeters at atmospheric pressure.
- Stagnation temperature remains the same across the shock wave, but the stagnation pressure and density decrease in the same ratio.
- Any stagnation process (e.g. Pitot tube) must induce a shock wave in front of it, if it is inserted into a supersonic flow

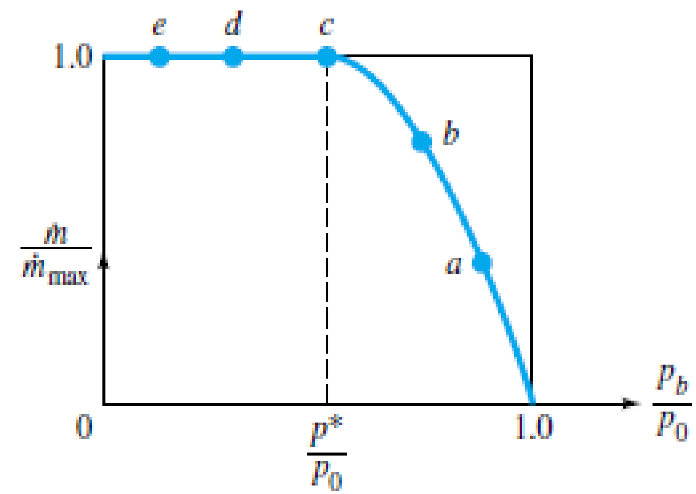
Operation of Converging and Diverging Nozzles

❑ Converging Nozzle

- Consider the converging nozzle sketched in Fig. Where, T_0 and p_0 (stagnation properties), p_e - exit pressure and p_b - back pressure.

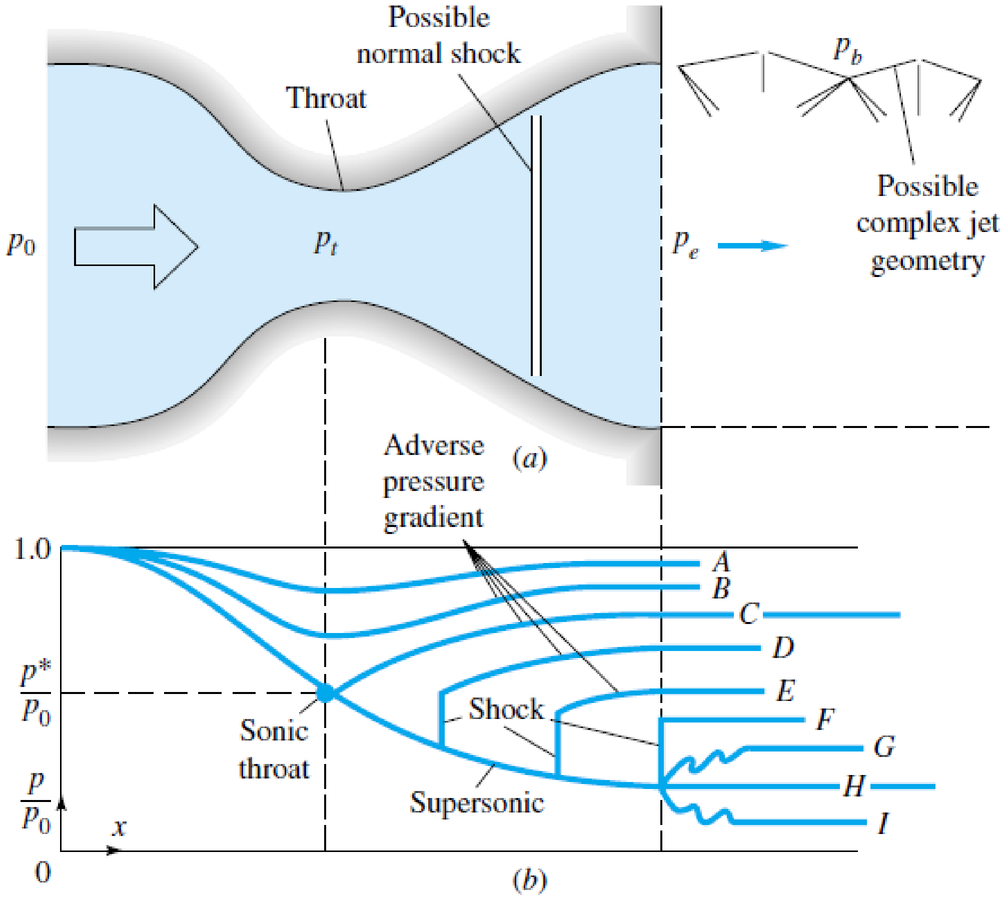


(b) *Pressure distribution caused by various back pressures*.



Mass flow versus back pressure.

❑ Converging-Diverging Nozzle operation



(b) Pressure distribution caused by various back pressures

A. Subsonic, possibly incompressible flow
B. Subsonic, compressible flow
C. Flow in the throat just becomes sonic, then returns to subsonic. Flow rate reached maximum, $A_e/A_t = A_e/A^*$
D. Normal shock after the throat
E. Normal shock moved downstream
F. Normal shock reached the exhaust plane
G. Over - expanded jet
H. Supersonic flow (design back pressure)
I. Under - expanded jet.

- *A nozzle when operating below the design value is said to be under-expanding and above the design pressure is called over - expanding*

EXAMPLE 9.9

A converging-diverging nozzle (Fig. 9.12*a*) has a throat area of 0.002 m^2 and an exit area of 0.008 m^2. Air stagnation conditions are $p_0 = 1000$ kPa and $T_0 = 500$ K. Compute the exit pressure and mass flow for (*a*) design condition and the exit pressure and mass flow if (*b*) $p_b \approx$ 300 kPa and (*c*) $p_b \approx$ 900 kPa. Assume $k = 1.4$.

Solution: *see White, page 602 - 603*

Fanno line and Rayleigh line

- So far we have limited our consideration mostly to isentropic flow (no heat transfer and no irreversibility such as friction)
- Actual fluid flows are generally non-isentropic. An important example of ***non-isentropic flow*** involves adiabatic flow with friction.
- Friction must be included for flow through long ducts, especially if the cross-sectional area is small.
- ***Fanno flow*** – is an ***adiabatic*** flow of an ideal gas through a *constant area duct* ***with friction.***
- ***Rayleigh flow*** - is frictionless constant area duct flow with ***heat transfer.***

- Combining the ***conservation of mass and energy equation*** relations into a single equation and plot it on an ***h-s diagram***, using property relations. The resultant curve is called ***the Fanno line.***

Property relations for Fanno Flows:

- Recalling that, $$\frac{T_{01}}{T_1} = 1 + \left(\frac{k-1}{2}\right)\mathrm{Ma}_1^2 \quad \text{and} \quad \frac{T_{02}}{T_2} = 1 + \left(\frac{k-1}{2}\right)\mathrm{Ma}_2^2$$

- Dividing the first equation by the second one and noting that $T_{01} = T_{02}$, we have: $$\frac{T_2}{T_1} = \frac{1 + \mathrm{Ma}_1^2(k-1)/2}{1 + \mathrm{Ma}_2^2(k-1)/2} \quad \text{……………………. (a)}$$

- From the ideal-gas equation of state,

$$\rho_1 = \frac{P_1}{RT_1} \quad \text{and} \quad \rho_2 = \frac{P_2}{RT_2} \quad \text{……………... (b)}$$

- Substituting these into the conservation of mass relation $\rho_1 V_1 = \rho_2 V_2$ and noting that, Ma = V/a and , $a = \sqrt{kRT}$, we have

$$\frac{T_2}{T_1} = \frac{P_2 V_2}{P_1 V_1} \quad OR \quad \frac{T_2}{T_1} = \frac{P_2 * Ma_2 * a_2}{P_1 * Ma_1 * a_1} \quad OR \quad \frac{T_2}{T_1} = \frac{P_2 * Ma_2 * \sqrt{kRT_2}}{P_1 * Ma_1 * \sqrt{kRT_1}}$$

$$OR \quad \frac{T_2}{T_1} = \frac{P_2 * Ma_2 * \sqrt{T_2}}{P_1 * Ma_1 * \sqrt{T_1}} \quad OR \quad \frac{T_2}{T_1}\left(\frac{\sqrt{T_1}}{\sqrt{T_2}}\right) = \frac{P_2 Ma_2}{P_1 Ma_1}$$

$$OR \quad \frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^2 \left(\frac{Ma_2}{Ma_1}\right)^2 \quad \text{...} \quad (c)$$

- Combining Eqs. (a) and (c) gives the pressure ratio across the shock as:

$$\boxed{\frac{P_2}{P_1} = \frac{Ma_1\sqrt{1 + Ma_1^2(k-1)/2}}{Ma_2\sqrt{1 + Ma_2^2(k-1)/2}}} \quad \text{..............................} \quad (d)$$

- Thus, Eq. (d) is a combination of the conservation of mass & energy equations; also called the equation of the Fanno line for an ideal gas with constant specific heats.
- ***Fanno line*** is the locus of states that have the same value of stagnation enthalpy and mass flux (mass flow per unit flow area).

- **Likewise,** combining the ***conservation of mass and momentum equations*** into a single equation and plotting it on ***the h-s diagram*** yield a curve called the ***Rayleigh line*.**

Property relations for Rayleigh Flows:

- Recalling the momentum equation as: $P_1 - P_2 = \rho_2 V_2^2 - \rho_1 V_1^2$

  And, $\rho V^2 = PkMa^2$

- After certain manipulation, we obtain

$$\boxed{\frac{P_2}{P_1} = \frac{1 + kMa_1^2}{1 + kMa_2^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(e)}$$

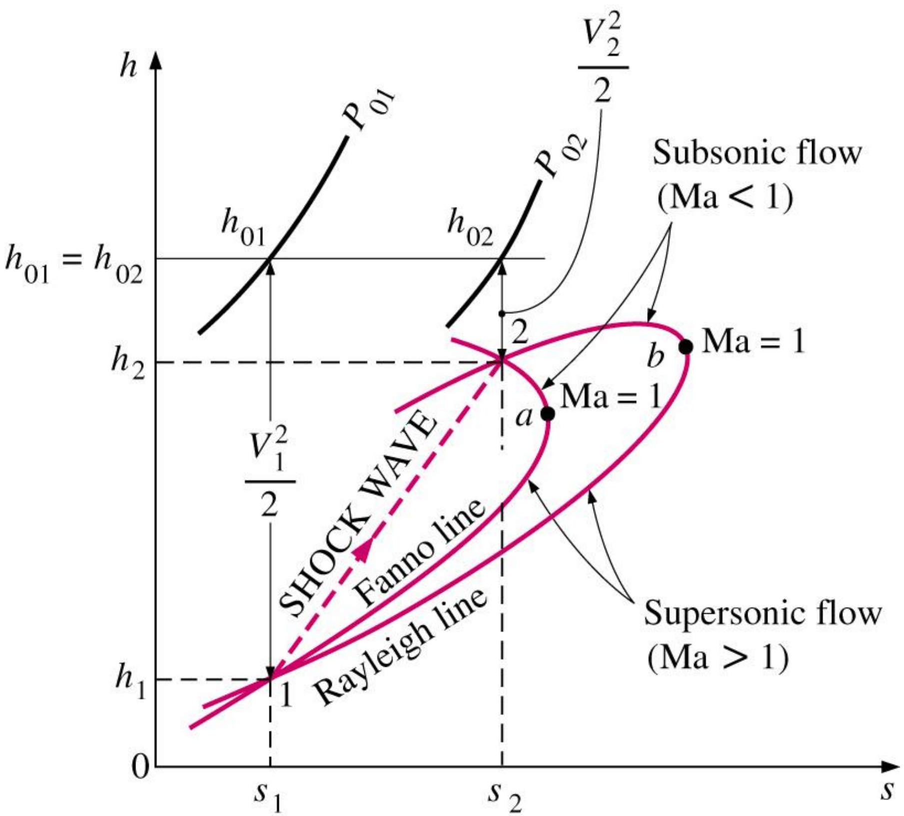
- Eq. (e) is the *Rayleigh line equation.*

- Combining Eqs. (d) and (e) yields

$$\boxed{Ma_2^2 = \frac{Ma_1^2 + 2/(k-1)}{2Ma_1^2 k/(k-1) - 1}} \quad \ldots\ldots\ldots\ldots \text{(f)}$$

- Eq. (f) represents the *intersections of the Fanno and Rayleigh lines* and relates the *Mach number upstream of the shock to that downstream of the shock.*

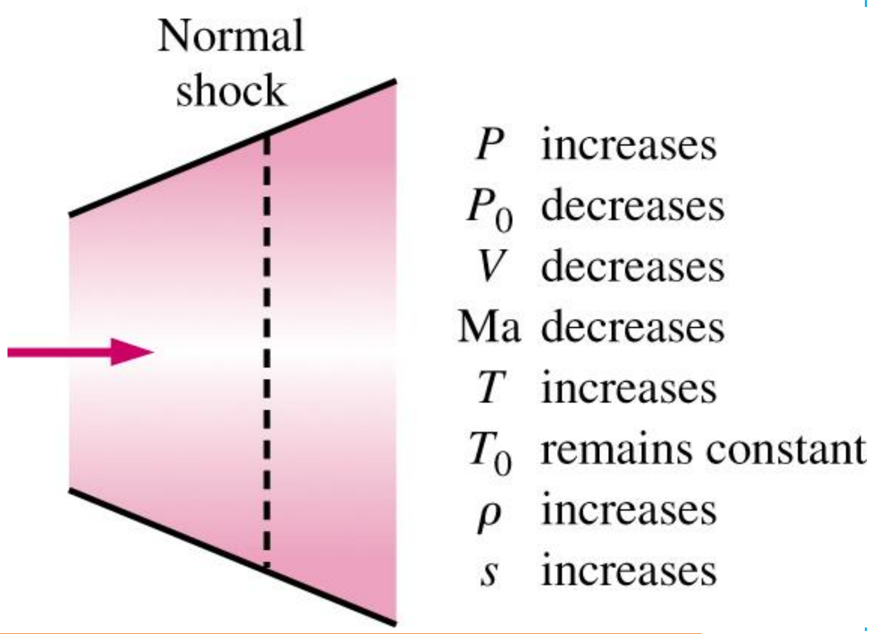
Fanno & Rayleigh lines plots on h-s diagram



- There are 2 points where the Fanno and Rayleigh lines intersect : points where all 3 conservation equations are satisfied
  - **Point 1:** before the shock (supersonic)
  - **Point 2:** after the shock (subsonic)
- The larger Ma is before the shock, the stronger the shock will be.
- Therefore the flow must change from supersonic to subsonic if a shock is to occur.
- Entropy increases from point 1 to point 2 : expected since flow through the shock is adiabatic but irreversible

- Equation for the Fanno line for an ideal gas with constant specific heats can be derived

$$\frac{P_2}{P_1}=\frac{Ma_1\sqrt{1+Ma_1^2(k-1)/2}}{Ma_2\sqrt{1+Ma_2^2(k-1)/2}}$$



Variation of flow properties across a normal shock [Rayleigh flow]

- *The **heat addition** causes a decrease in stagnation pressure, which is known as the Rayleigh effect and is critical in the design of combustion systems.*
- ***Heat addition** will cause both supersonic and subsonic Mach numbers to approach Mach 1.*
- Conversely, ***heat rejection*** decr-eases a subsonic Ma no. and increases a supersonic Ma no. along the duct.
- *It can be shown that, for calorically perfect flows the maximum entropy occurs at Ma = 1.*

The effects of friction on the properties of Fanno flow

Property	Subsonic	Supersonic
Velocity, V	Increase	Decrease
Mach number, Ma	Increase	Decrease
Stagnation temperature, T_0	Constant	Constant
Temperature, T	Decrease	Increase
Density, ρ	Decrease	Increase
Stagnation pressure, P_0	Decrease	Decrease
Pressure, P	Decrease	Increase
Entropy, s	Increase	Increase

The effects of heating and cooling on the properties of Rayleigh flow

	Heating		***Cooling***	
Property	Subsonic	Supersonic	Subsonic	Supersonic
Velocity, V	Increase	Decrease	Decrease	Increase
Mach number, Ma	Increase	Decrease	Decrease	Increase
Stagnation temperature, T_0	Increase	Increase	Decrease	Decrease
Temperature, T	Increase for Ma $< 1/k^{1/2}$ Decrease for Ma $> 1/k^{1/2}$	Increase	Decrease for Ma $< 1/k^{1/2}$ Increase for Ma $> 1/k^{1/2}$	Decrease
Density, ρ	Decrease	Increase	Increase	Decrease
Stagnation pressure, P_0	Decrease	Decrease	Increase	Increase
Pressure, P	Decrease	Increase	Increase	Decrease
Entropy, s	Increase	Increase	Decrease	Decrease

Two-Dimensional Supersonic Flow

- Up to this point we have considered only one-dimensional compressible-flow theories. But, now we:
  - Extend one-dimensional supersonic flow to two dimensions
  - Derive Mach Cone relations
  - Analyze oblique shock wave
  - Explain the shape of shock wave around an obstacle in supersonic flow
  - Illustrate consequences to supersonic body shape design

Sound Wave Propagation from a Moving Source

- As the particle moves, it continually crashes against fluid particles and sends out spherical sound waves emanating from every point along its path.
- This radiated pattern of the pressure wave is seen to be a cone, called the ***Mach cone***.
- The semi-angle of the cone is called the ***Mach angle,μ*** and it is given by:

$$\mu = \sin^{-1}\frac{a\,\delta t}{U\,\delta t} = \sin^{-1}\frac{a}{U} = \sin^{-1}\frac{1}{\mathrm{Ma}}$$

- In a 2D flow, the cone reduces to a pair of intersecting lines each of which is called ***Mach line or Mach wave.***

Sound wave propagation from a moving source

- As the particle moves, it continually crashes against fluid particles and sends out spherical sound waves emanating from every point along its path.

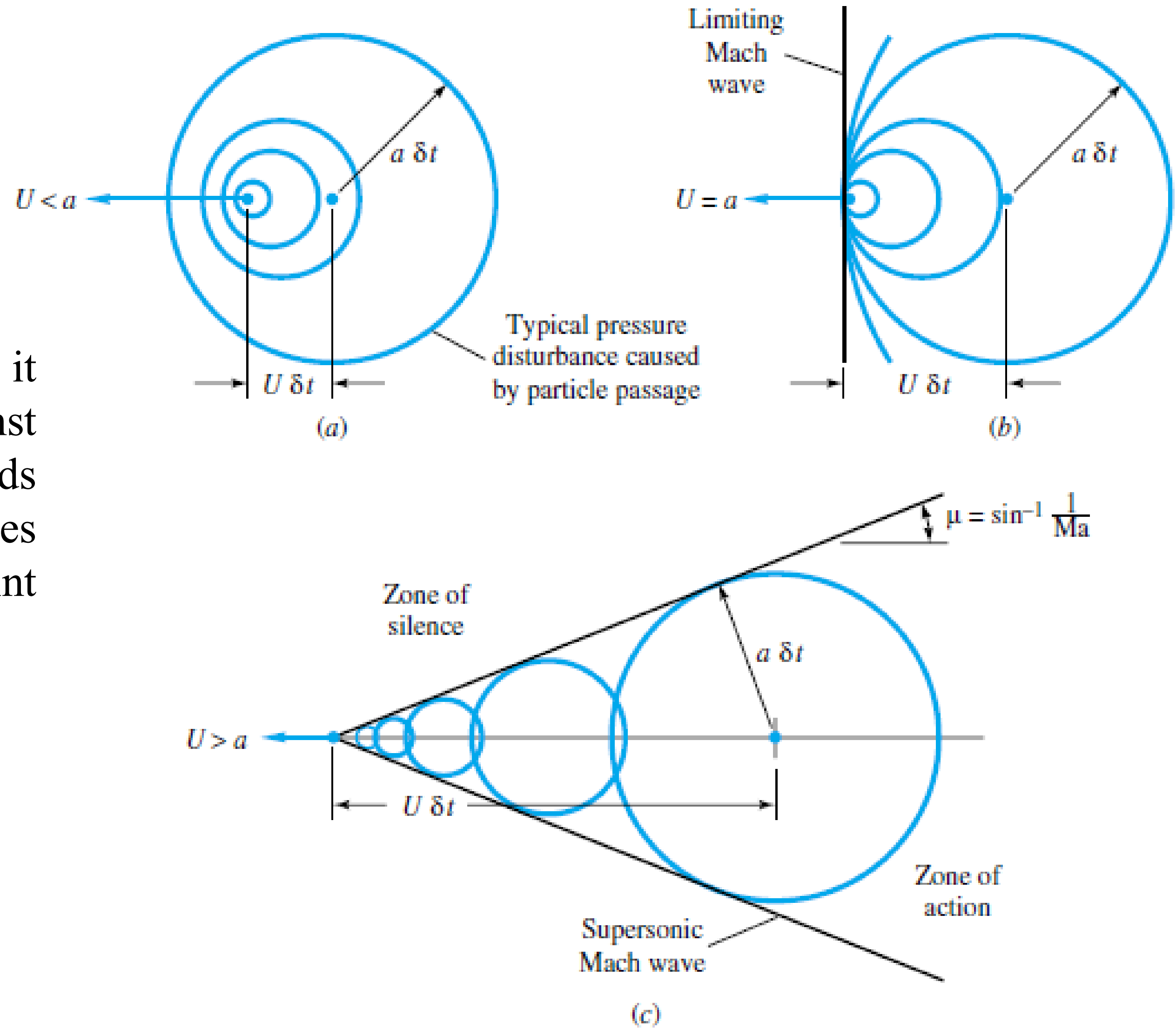


Fig. 9.18 Wave patterns set up by a particle moving at speed U into still fluid of sound velocity a: (a) subsonic, (b) sonic, and (c) supersonic motion.

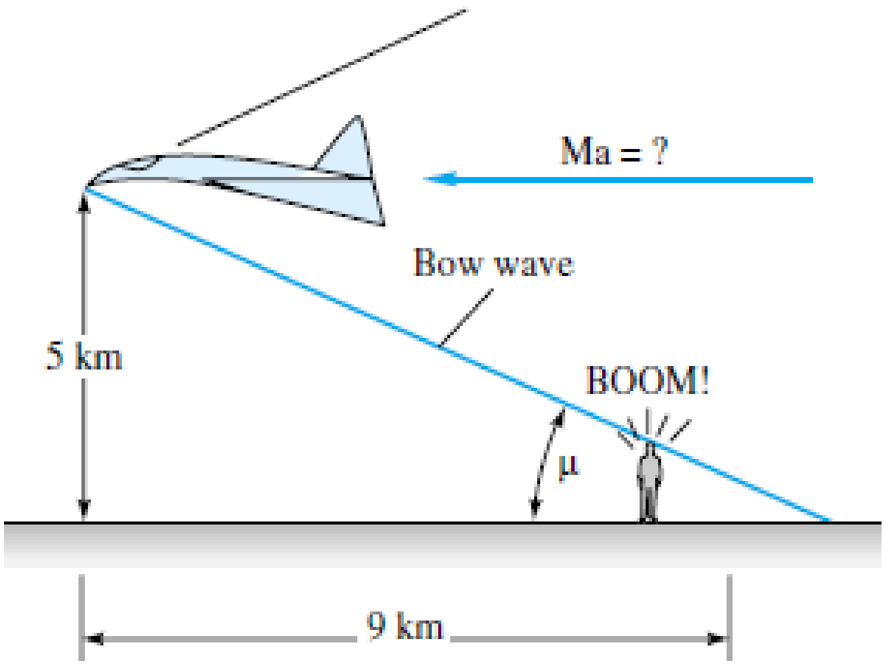
- The pressure waves do not reach the region to the left of the normal front and this region is called ***the silence zone*** (out side the cone)
- In supersonic flow the pressure disturbance are felt only ***inside the cone*** and this region is called ***zone of action.***
- It is precisely due to this reason that an airplane moving at supersonic speed is *not heard by stationary observer until the plane has passed him.*

EXAMPLE

An observer on the ground does not hear the sonic boom caused by an airplane moving at 5-km altitude until it is 9 km past her. What is the approximate Mach number of the plane? Assume a small disturbance and neglect the variation of sound speed with altitude.

Solution

A finite disturbance like an airplane will create a finite-strength oblique-shock wave whose angle will be somewhat larger than the Mach-wave angle μ and will curve downward due to the variation in atmospheric sound speed. If we neglect these effects, the altitude and distance are a measure of μ, as seen in Fig. Thus



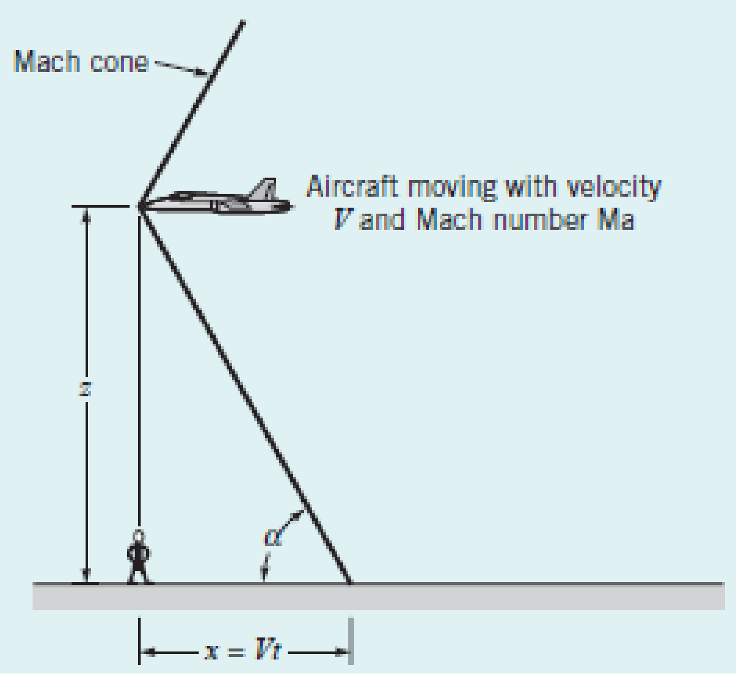
$$\tan \mu = \frac{5 \text{ km}}{9 \text{ km}} = 0.5556 \qquad \text{or} \qquad \mu = 29.05°$$

$$\text{Ma} = \frac{1}{\sin \mu} = 2.06$$

Example 2

- An aircraft cruising at 1000-m elevation, above you moves past in a flyby. It is moving with a Mach number equal to 1.5, speed of sound 343.3 m/s and the ambient temperature is 20 °C. **Find,** How many seconds after the plane passes overhead do you expect to wait before you hear the aircraft?

- **Solution:**



$$\alpha = \tan^{-1}\frac{z}{x} = \tan^{-1}\frac{1000}{Vt}$$

$$\mathrm{Ma} = \frac{1}{\sin\alpha}$$

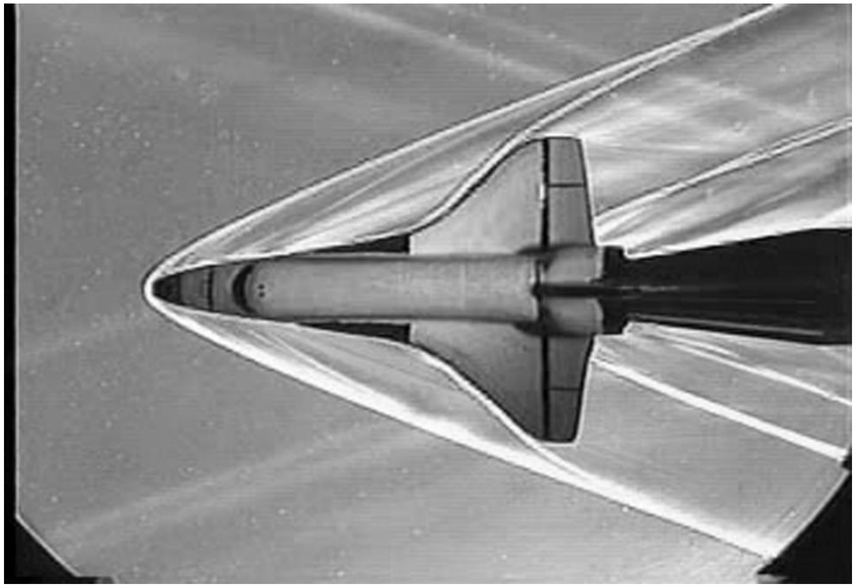
$$\mathrm{Ma} = \frac{1}{\sin\left[\tan^{-1}(1000/Vt)\right]}$$

Using Ma = 1.5, we get

$$1.5 = \frac{1}{\sin\left\{\tan^{-1}\left[\frac{1000\ \mathrm{m}}{(1.5)(343.3\ \mathrm{m/s})t}\right]\right\}}$$

$$t = 2.17\ \mathrm{s}$$

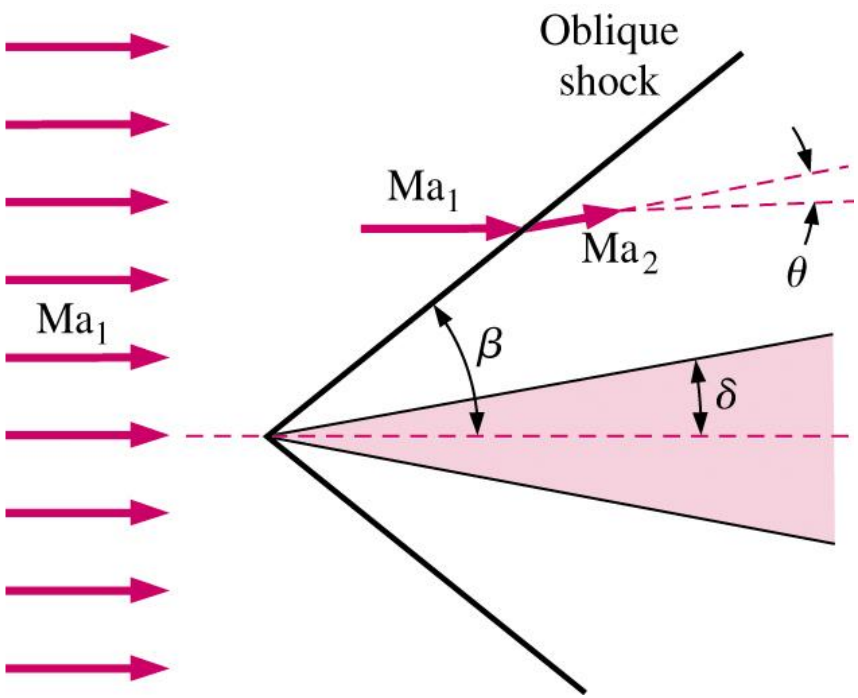
Oblique Shock Waves



- Not all shocks are normal to flow direction.
- Some are inclined to the flow direction, and are called **oblique shocks**

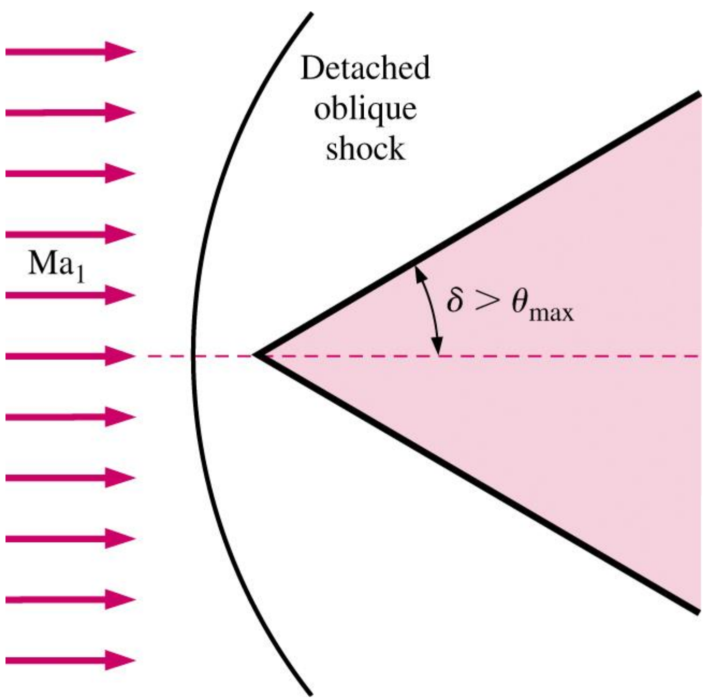
Reading Assignment

- *Oblique shock waves: characteristic features, governing equations, calculation of properties*
- *Textbook : White and Manson*



- At leading edge, flow is deflected through an angle θ called the ***turning/deflection angle***
- Result is a straight oblique shock wave aligned at **shock angle** β relative to the flow direction
- Due to the displacement thickness, θ is slightly greater than the wedge half-angle δ.

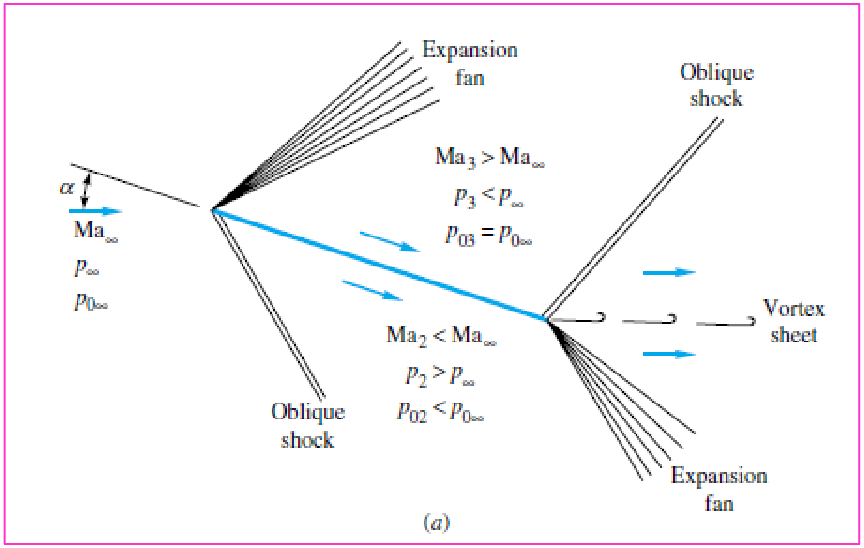
- Like normal shocks, *Ma decreases across the oblique shock*, and are *only possible if upstream flow is supersonic*
- However, unlike normal shocks in which the downstream Ma is always subsonic, Ma_2 of an oblique shock can be subsonic, sonic, or supersonic depending upon Ma_1 and θ.

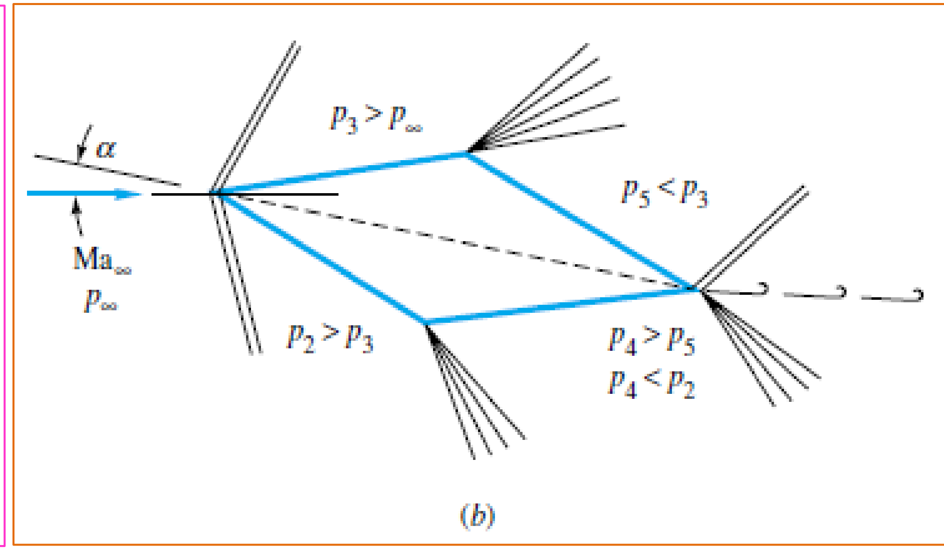


- ✓ If wedge half angle $\theta > \theta_{max}$, a detached oblique shock or bow wave is formed
- ✓ Much more complicated that straight oblique shocks.
- ✓ Requires CFD for analysis.

Lift & Drag on Supersonic Airfoil

- **Shock expansion theory:** a very successful application theory for supersonic airfoil, which is the combination of ***oblique-shock and Prandtl-Meyer expansion theories.***





Flat plate and a diamond-shaped foil

- **Fig. *a*** shows a flat-plate foil at an angle of attack. There is a LE shock on the lower edge with flow deflection, $\theta = \alpha$ while the upper edge has an expansion fan with increasing Prandtl-Meyer angle $\Delta\omega = \alpha$.

- We compute, *P_3 with expansion theory and p_2 with oblique-shock theory*.
- Force on the plate is: $F = (p_2 - p_3)Cb$
- Lift force normal to the stream is:

$$L = F\cos\alpha \text{ and the}$$

- Drag parallel to the stream is:

$$D = F\sin\alpha$$

> Where C is the chord length and b the span width (assuming no wingtip effects).
> F is normal to the plate,

> Introducing perfect-gas-law identity: $\frac{1}{2}\rho V^2 \equiv \frac{1}{2}kp\,\mathrm{Ma}^2$

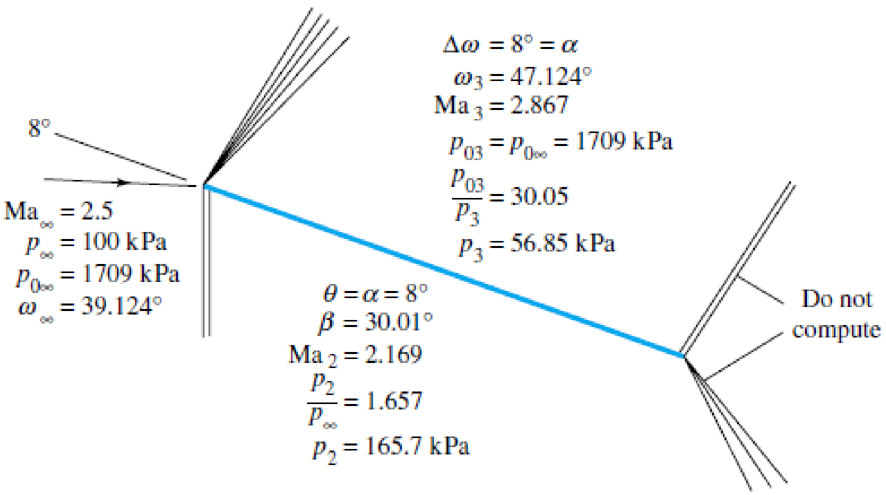
$$C_L = \frac{L}{\frac{1}{2}kp_\infty \mathrm{Ma}_\infty^2\, bC} \qquad C_D = \frac{D}{\frac{1}{2}kp_\infty \mathrm{Ma}_\infty^2\, bC}$$

Note: the typical supersonic C_L is much smaller than the subsonic value $C_L \approx 2\pi\alpha$, but the lift can be very large because of the large value of $\boldsymbol{1/2\rho V^2}$ at supersonic speeds.

EXAMPLE 9.19

A flat-plate airfoil with C = 2 m is immersed at α = 8° in a stream with Ma_∞ = 2.5 and p_∞ = 100 kPa. Compute (*a*) C_L and (*b*) C_D, and compare with low-speed airfoils. Compute (*c*) lift and (*d*) drag in newtons per unit span width.

Solution



$$F = (p_2 - p_3)bC$$

$$= (165.7 - 56.85)(\text{kPa})(1\text{ m})(2\text{ m})$$

$$= 218 \text{ kN}$$

The lift and drag per meter width are thus

$$L = F \cos 8° = 216 \text{ kN}$$

$$D = F \sin 8° = 30 \text{ kN}$$

These are very large forces for only 2 m^2 of wing area.

$$C_L = \frac{216 \text{ kN}}{\frac{1}{2}(1.4)(100 \text{ kPa})(2.5)^2(2 \text{ m}^2)} = 0.246$$

$$C_D = \frac{30 \text{ kN}}{\frac{1}{2}(1.4)(100 \text{ kPa})(2.5)^2(2 \text{ m}^2)} = 0.035$$

Thank you

End